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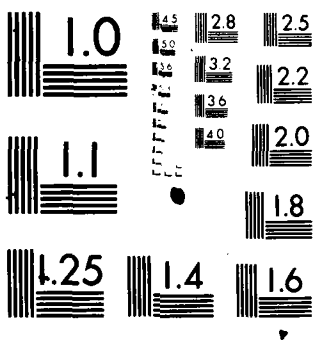

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THE LINEAR FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL SINGULAR BOUNDARY VALUE PROBLEM

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ABSTRACT

The following model problem is studied:

$$\Omega \;:\; -[\frac{1}{r}\frac{\partial}{\partial r}(r\beta\frac{\partial u}{\partial r}) + \frac{\partial}{\partial z}(\beta\frac{\partial u}{\partial z})] = f$$

$$\Gamma_1 \;:\; u = 0$$

where Ω is a bounded open domain with $r < 0$ in (r,z) plane, $\Gamma_1 = \partial\Omega\backslash\Gamma o$, $\Gamma o = \partial\Omega \cap \{(r,z) : r = 0\}$. We introduce weighted Sobolev spaces $V^k(k = 1,2)$, and prove:

(1) The problem has a unique solution u, and $u \in V_o^1(\Omega) \cap V^2(\Omega)$.

(2) The linear finite element solution u_h exists and is unique.

(3) The error $u-u_h$ in "energy norm" is of $0(h^2)$. Particularly, if Ω is a polygon, then

$$\|u - u_h\|_{1,\Omega} = 0(h)$$

$$\|u - u_h\|_{0,\Omega} = 0(h^2)$$

where $\|\cdot\|_{k,\Omega}(k = 1,2)$ are the V^k norms.

AMS (MOS) Subject Classifications: 65N30, 65N15

Key words: Finite element method; two dimensional singular boundary value problem; weighted Sobolev spaces; order of convergence.

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SIGNIFICANCE AND EXPLANATION

For two dimensional singular boundary value problems of form:

$$\Omega : \frac{\partial^2 u}{\partial x^2} + \frac{k}{y}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Gamma_1 : u = g$$

where Ω is a bounded open domain with $y > 0$ in (x,y)-plane, $\Gamma_1 = \partial\Omega \cap \{(x,y) : y > 0\}$, Parter [13] has proposed finite difference methods and established the corresponding theory. Wilson [16] has proposed a finite element method for other types of two dimensional singular problems, but did not study the convergence theory. This paper extends earlier works on convergence of the methods to such problems.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE LINEAR FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL SINGULAR BOUNDARY VALUE PROBLEM

S. Z. Zhou

Introduction

The numerical solution of singular boundary value problems have been studied by several authors. The finite difference methods and its theory for a type of two-dimensional singular boundary value problems are given in [10], [13]. The finite element method for axisymmetric elastic solid is proposed in [16]. [5], [11], [14] and [20], gives a proof of the convergence of the finite element methods for one dimensional singular problems. [12] proves the "optimal" order of convergence for the method of [16] provided the loads are axisymmetric and the solution is in $C^{k+1}(\bar{\Omega})$. The convergence of the linear finite element method for two dimensional singular Dirichlet problem is proved in [18]. In this paper we will prove the so-called "optimal" order of convergence of the linear finite element method for the following model problem:

$$\begin{aligned} \Omega :\quad & -\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\beta \frac{\partial u}{\partial r}\right\} + \frac{\partial}{\partial z}\left\{\beta \frac{\partial u}{\partial z}\right\}\right] = f \\ \Gamma_1 :\quad & u = 0 \end{aligned} \tag{1.1}$$

where Ω is a bounded open domain with $r > 0$ in (r,z)-plane, $\Gamma_1 = \partial\Omega/\Gamma_0$, $\Gamma_0 = \partial\Omega \cap \{(r,z) : r = 0\}$.

We assume:

(i) The function β is uniformly Lipschitz continuous in Ω.

(ii) $\beta \geq \beta_0 > 0$, β_0 is a constant.

(iii) $r^{1/2} f \in L^2(\Omega)$.

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Under the (x,y,z) coordinate system we have

$$-\int_{\Omega^*} v_n^* \frac{\partial \phi^*}{\partial r} r^{-1} dxdydz = \int_{\Omega^*} \left(\frac{\partial v_n^*}{\partial x} \cos\theta + \frac{\partial v_n^*}{\partial y} \sin\theta\right) \phi^* r^{-1} dxdydz \tag{2.5}$$

Since $r^{-1}\phi^*$ and $r^{-1}\frac{\partial \phi^*}{\partial r}$ are bounded in Ω^*, $v^* \in H^1(\Omega^*)$, we may take the limit through (2.5) as $n \to \infty$ and hence we obtain (2.5) as well as (2.4) with v, v^* replacing v_n, v_n^* respectively. (2.1) is proved. Q.E.D.

Simple calculation derives the following results.

<u>Corollary 2.1</u>. If $v^* \in H^1(\Omega^*)$, then

$$\frac{\partial v^*}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r},$$

$$\frac{\partial v^*}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \qquad \text{in } \Omega^* .$$

<u>Corollary 2.2</u>. Assume v is independent of θ. Then we have for $v^* \in H^1(\Omega^*)$:

$$\frac{\partial v^*}{\partial x} = \frac{\partial v}{\partial r}\cos\theta , \quad \frac{\partial v^*}{\partial y} = \frac{\partial v}{\partial r}\sin\theta ;$$

for $v^* \in H^2(\Omega^*)$

$$\frac{\partial^2 v^*}{\partial x^2} = \frac{\partial^2 v}{\partial r^2}\cos^2\theta + \frac{\partial v}{\partial r}\frac{\sin^2\theta}{r} , \quad \frac{\partial^2 v^*}{\partial y^2} = \frac{\partial^2 v}{\partial r^2}\sin^2\theta + \frac{\partial v}{\partial r}\frac{\cos^2\theta}{r}$$

$$\frac{\partial^2 v^*}{\partial x \partial y} = \left(\frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r}\right)\sin\theta\cos\theta ,$$

$$\frac{\partial^2 v^*}{\partial z^2} = \frac{\partial^2 v}{\partial z^2} , \quad \frac{\partial^2 v^*}{\partial x \partial z} = \frac{\partial^2 v}{\partial r \partial z}\cos\theta , \quad \frac{\partial^2 v^*}{\partial y \partial z} = \frac{\partial^2 v}{\partial r \partial z}\sin\theta .$$

3. Spaces V^1, V^2 ([2], [17])

We define functionals $\|\cdot\|_{k,\Omega}$, $k = 0, 1, 2$, as follows:

$$\|v\|_{0,\Omega} = \left(\int_\Omega v^2 r\,dr\,dz\right)^{1/2}$$

$$\|v\|_{1,\Omega} = \left(\sum_{|\alpha|\leq 1} \|\partial^\alpha v\|^2_{0,\Omega}\right)^{1/2}$$

$$\|v\|_{2,\Omega} = \left(\sum_{|\alpha|\leq 2} \|\partial^\alpha v\|^2_{0,\Omega} + \left\|\frac{1}{r}\frac{\partial v}{\partial r}\right\|^2_{0,\Omega}\right)^{1/2}$$

Definition 3.1. Assume that D is an open or closed set in (r,z)-plane, D^* the correspondant axisymmetric set in (x,y,z)-space, $A(D)$ the set of real functions defined in D, and

$$A^*(D^*) = \{v^* : v^* \text{ real function defined in } D^*\text{, and there exists } v \in A(D) \text{ such that } v^*(x,y,z) = v(\sqrt{x^2+y^2}, z)\}.$$

We define a mapping $T: A^*(D^*) \to A(D)$ as follows:

$$Tv^*(x,y,z) = v(r,z)$$

Obviously, the mapping T is one-to-one.

Definition 3.2. $U^k(\Omega^*) = H^k(\Omega^*) \cap A^*(\Omega^*)$, $k = 0,1,2$.

It is easy to see that $U^k(\Omega^*)$ is a closed subspace in $H^k(\Omega^*)$. Now establish the relations between the norms $\|\cdot\|_{H^k(\Omega^*)}$ and the functionals $\|\cdot\|_{k,\Omega}$ for the elements of $U^k(\Omega^*)$.

Lemma 3.1. Assume $v^* \in U^k(\Omega^*)$, $v = Tv^*$. then $\|v\|_{k,\Omega} < \infty$, and

$$\|v^*\|^2_{H^k(\Omega^*)} = 2\pi\|v\|^2_{k,\Omega}, \quad \forall u^* \in U^k(\Omega^*),\ k = 0,1\ . \tag{3.1}$$

$$\frac{3\pi}{2}\|v\|^2_{2,\Omega} \leq \|v^*\|^2_{H^2(\Omega^*)} \leq 2\pi\|v\|^2_{2,\Omega}, \quad \forall u^* \in U^2(\Omega^*) \tag{3.2}$$

Proof. By direct computation and corollary 2.2.

<u>Definition 3.3</u>. $V^k(\Omega) = \{v: v = Tv^*,\ v^* \in U^k(\Omega^*)\}$, $k = 0, 1, 2$.

It follows from lemma 3.1 and the closeness of $U^k(\Omega^*)$ in $H^k(\Omega^*)$ that $V^k(\Omega)$, $k = 0, 1, 2$, are Banach spaces. We need the following subspace $V_0^1(\Omega)$ of $V^1(\Omega)$:

$$V_0^1(\Omega) = \{v: v = Tv^*,\ v^* \subset U^1(\Omega^*) \cap H_0^1(\Omega^*)\}.$$

Let $v \in V_0^1(\Omega)$, $v = Tv^*$, $\operatorname{tr} v^*$ be the trace of v^* on $\partial\Omega^*$. We define $T(\operatorname{tr} v^*)$ as the trace of v on Γ_1. Obviously, it is zero.

By lemma 3.1 and the embedding theorems of $H^k(\Omega^*)$. We obtain the correspondent theorems of $V^k(\Omega^*)$. Particularly, we have the following result.

<u>Lemma 3.2</u>. There exists a constant C' such that

$$\|v\|_{1,\Omega}^2 \le C' \int_\Omega [(\frac{\partial v}{\partial r})^2 + (\frac{\partial v}{\partial z})^2] r\,dr\,dz,\ \forall\ v \in V_0^1(\Omega) \qquad (3.3)$$

Finally, the following statement on denseness may be proved (see [17] for V^1. the proof is similar for V^2).

<u>Lemma 3.3</u>. Assume that the domain Ω has a locally Lipschitz Boundary. Then $C^\infty(\bar{\Omega})$ is dense in $V^k(\Omega)$, $k = 1, 2$.

<u>Remark 3.1</u>. Lemma 3.3 is not a direct corollary of the denseness theorem of $H^k(\Omega^*)$. If $v^* \in H^k(\Omega^*)$, then there exists a sequence $v_n^* \in C^\infty(\bar{\Omega}^*)$ converging to v^* in $H^k(\Omega^*)$. But we can not claim that $v_n^* \in A^*(\Omega^*)$.

<u>Remark 3.2</u>. The facts $v \in C^\infty(\bar{\Omega})$ and $v = Tv^*$ do not imply that $v^* \in C^\infty(\bar{\Omega}^*)$. Counter example: $v = r$. But $v \in C^0(\bar{\Omega}) \iff v^* \in C^0(\bar{\Omega}^*)$.

4. Solution of problem (1.1)

We define a bilinear form $B(\cdot,\cdot)$ on $V^1(\Omega) \times V^1(\Omega)$ and a linear functional $F(\cdot)$ on $V^1(\Omega)$ as follows:

$$B(u,v) = \int_\Omega \beta\left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) r\,dr\,dz$$

$$F(v) = \int_\Omega f v r\,dr\,dz$$

Then we have the variational formulation of problem (1.1): Find $u \in V_0^1(\Omega)$ such that

$$B(u,v) = F(v), \quad \forall\, v \in V_0^1(\Omega) \tag{4.1}$$

From now on we assume that Ω has a locally Lipschitz boundary.

Theorem 4.1 Problem (4.1) has a unique solution.

Proof: It follows from lemma 3.2 and assumptions (i)-(ii) that the bilinear form $B(u,v)$ is coercive and continuous on $V_0^1(\Omega) \times V_0^1(\Omega)$. And the linear functional $F(v)$ is continuous on $V_0^1(\Omega)$ by virtue of assumption (iii). Hence the conclusion of the theorem is a result of the Lax-Milgram theorem. Q.E.D.

Remark 4.1. Let u be the solution of problem (4.1). Since $B(u.v)$ is symmetric, U is also the solution of the following problem: Find $u \in V_0^1(\Omega)$ such that

$$J(u) = \min_{v \in V_0^1(\Omega)} J(v)$$

where $J(v) = B(v,v) - 2F(v)$.

Consider the boundary value problem in Ω^* corresponding to problem (1.1):

$$\begin{aligned} &\Omega^*: \; -\left[\frac{\partial}{\partial x}\left(\beta^*\frac{\partial w^*}{\partial x}\right) + \frac{\partial}{\partial y}\left(\beta^*\frac{\partial w^*}{\partial y}\right) + \frac{\partial}{\partial z}\left(\beta^*\frac{\partial w^*}{\partial z}\right)\right] = f^* \\ &\partial\Omega^*: \; w^* = 0 \end{aligned} \tag{4.2}$$

where $\beta^* = T^{-1}\beta$, $f^* = T^{-1}f$. Correspondant variational problem is: Find $w^* \in H_0^1(\Omega^*)$ such that

$$B_1(w^*,v^*) = F_1(v^*), \quad \forall\, v^* \in H_0^1(\Omega^*) \tag{4.3}$$

where

$$B_1(w^*,v^*) = \int_{\Omega^*} \beta^*(\frac{\partial w^*}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w^*}{\partial y}\frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}\frac{\partial v^*}{\partial z})dxdydz$$

$$F_1(v^*) = \int_{\Omega^*} f^*v^*dxdydz$$

From now on we assume:

(iv). The boundary $\partial\Omega^*$ is smooth enough to ensure that problem (4.3) has a unique solution w^* and $w^* \in H_0^1(\Omega^*) \cap H^2(\Omega^*)$. For example, we may assume that $\partial\Omega^*$ is of class C^2 (see, for instance, [9, p.176]) or that the domain Ω^* is convex.

__Theorem 4.2__. Let u be the solution of problem (4.1). Then

$$u \in V_0^1(\Omega) \cap V^2(\Omega)$$

__Proof__: Let $u^* = T^{-1}u$. We define for $v^* \in H_0^1(\Omega^*)$ that

$$v(r,\theta,z) = v^*(x,y,z)$$

$$\tilde{v}(r,z) = \int_0^{2\pi} v(r,\theta,z)d\theta$$

It is easily proved that

$$\tilde{v} \in V_0^1(\Omega) \tag{4.4}$$

$$\frac{\partial\tilde{v}}{\partial r} = \int_0^{2\pi}\frac{\partial v}{\partial r}\,d\theta,\ \frac{\partial\tilde{v}}{\partial z} = \int_0^{2\pi}\frac{\partial v}{\partial z}\,d\theta \tag{4.5}$$

Now we prove that u^* is the solution of problem (4.3). It follows from lemma 2.1, (4.4), (4.5) and (4.1) that for $v^* \in H_0^1(\Omega^*)$

$$B_1(u^*,v^*) - F_1(v^*) = \int_\Omega \left[\int_0^{2\pi}(\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} - fu)d\theta\right]rdrdz$$

$$= \int_\Omega \left[(\frac{\partial u}{\partial r}\frac{\partial\tilde{v}}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial\tilde{v}}{\partial z}) - f\tilde{v}\right]rdrdz = B(u,\tilde{v}) - F(\tilde{v}) = 0.$$

Hence u^* is the solution of problem (4.3), and $u^* \in H_0^1(\Omega^*)\quad H^2(\Omega^*)$ by assumption (iv). According to definition 3.3 we obtain the conclusion of the theorem. Q.E.D.

5. Linear finite element solution and its order of convergence order

Assume that the domain Ω is convex. Let $T_h = \{C_1,\dots,Cm\}$ be a triangulation of Ω, h_i the maximum edge of the triangle C_i, θ_i the minimum angle of C_i, $h = \max_i h_i$, $\theta = \min \theta_i$, $\Omega_h = \sum_{i=1}^{m} C_i$, assume:

(a). $\theta > \theta_o > 0$, θ_o is independent of h ([7], [19]).

Define a linear finite element space V^h as follows:

$$V^h = \{v_h \subset C^0(\bar{\Omega}) : v_h \text{ is linear function in } C_i,\ i = 1,\dots,m;$$
$$v_h = 0 \text{ on } (\Omega - \Omega_h) \cup \Gamma_1\} .$$

Then it is easy to prove that $V^h \subset V_0^1(\Omega)$. We have the correspondant discrete problem for problem (4.1): Find $u_h \in V^h$ such that

$$B(u_h,v_h) = F(v_h),\ \forall\ v_h \in V^h \tag{5.1}$$

Theorem 5.1. Problem (5.1) has a unique solution.

The proof is similar to that of theorem 4.1.

Remark 5.1. The solution U_h of (5.1) is also the solution of the minimization problem: Find $U_h \in V^h$ such that $J(U_h) = \min_{V_h \in V^h} J(V_h)$.

Assume that u is the solution of problem (4.1), U_I the piecewise linear intrpolation corresponding to the triangulation T_h. For any triangle $C \in T_h$, we now estimate $\|u - u_I\|_{1,C}$. Let $Pj = (r_j,z_j)$, $j = 1,2,3$ be the vertexes of C, $\lambda_j(r,z), j = 1,2,3$ the so-called barycentric coordinates ([4, p. 45]), i.e. the basis functions for the linear interpolcation on C:

$$\lambda_j(P_i) = \delta_{ij}\ (i,j = 1,2,3)$$

Then we have for any function v defined on c and its linear interpolation V_I:

$$\sum_j \lambda_j(P)v(P_j) = v_I(P),\ \forall\ P \in C, \tag{5.2}$$

Particularly,

$$\sum_j \lambda_j = 1,\ \sum_j \lambda_j r_j = r,\ \sum_j \lambda_j z_j = z,\ \forall (r,z) \in C, \tag{5.3}$$

It follows from (5.3) that

$$\sum_j \lambda_j(r_j - r) = \sum_j \lambda_j(z_j - z) = 0,\ \forall\ (r,z) \in C \tag{5.4}$$

The proof of the following lemma belongs to [3].

Lemma 5.1. Assume that $v \in V^2(C)$, and the condition (a) is true. Then

$$\|v - v_I\|^2_{1,c} \le Mh^2\|v\|^2_{2,c}, \tag{5.5}$$

where the constant M is independent of C and V.

Proof: Assume $v \in C^\infty(c)$ temporarily. Expand v at the point $P = (r,z)$ by using the Taylor's formula with integral remainder (see, for instance [6,p.36]):

$$v(P_j) - v(P) = (r_j - r)\frac{\partial v(P)}{\partial r} + \int_0^1 (1-t)d_j^2 v(M_j)dt, \quad j = 1,2,3 . \tag{5.6}$$

where

$$d_j = (r_j - r)\frac{\partial}{\partial r} + (Z_j - z)\frac{\partial}{\partial z}, \quad d_j^2 = d_j d_j$$

$$M_j = P_j t + P(1-t)$$

It follows from (5.2), (5.3) and (5.6) that

$$v_I(P) - v(P) = \sum_j \lambda_j(P)[v(P_j) - v(P)]$$

$$= \sum_j [\lambda_j(P)(r_j - r)\frac{\partial v(P)}{\partial z} + \lambda_j(P)(z_j - z)\frac{\partial v(P)}{\partial z}]$$

$$+ \sum_j \int_0^1 (1 - t)\lambda_j(P)d_j^2 v(M_j)dt$$

By virtue of (5.4) the first sum vanishes, and we have

$$v_I(P) - v(P) = \sum_j \int_0^1 (1 - t)\lambda_j(P)d_j^2 v(M_j)dt \tag{5.7}$$

Differentiating (5.7) we obtain

$$\frac{\partial v_I}{\partial r} - \frac{\partial v}{\partial r} = \sum_j \int_0^1 (1-t)(\frac{\partial \lambda_j}{\partial r} d_j^2 - 2\lambda_j d_j \frac{\partial}{\partial r})v(M_j)dt + \sum_j \int_0^1 (1-t)\lambda_j d_j^2[\frac{\partial v(M_j)}{\partial r}(1-t)]dt \tag{5.8}$$

Integrating by parts the integrals in the second sum, noting (5.4) and that $\frac{d}{dt}[d_j v(M_j)] = d_j^2 v(M_j)$, we derive from (5.8) that

$$\frac{\partial v_I}{\partial r} - \frac{\partial v}{\partial r} = \sum_j \int_0^1 (1 - t)\frac{\partial \lambda_j}{\partial r} d_j^2 v(M_j)dt \tag{5.9}$$

It follows from the uniform basis condition (a) that (see, for instance, [8] or [15, p.137])

$$|\frac{\partial \lambda_j}{\partial r}|, |\frac{\partial \lambda_j}{\partial z}| \le M_1 h^{-1} \tag{5.10}$$

where h is the maximum edge of c, $M_1 = 4/\sin\theta_0$. Hence we have

$$|\frac{\partial v_I}{\partial r} - \frac{\partial v}{\partial r}| \le M_1 h^{-1} \sum_j \int_0^1 (1-t)|d_j^2 v(M_j)|dt \ ,$$

and then

$$\int_c |\frac{\partial v_I}{\partial r} - \frac{\partial v}{\partial r}|^2 r dr dz \le M_1^2 h^{-2} \int_c \left(\sum_j \int_0^1 (1 - t)^{1/4} (1 - t)^{5/4} |d_j^2 v(M_j)| dt\right)^2 r dr dz$$

$$\le 3M_1^2 h^{-2} \sum_j \int_c \left(\int_0^1 (1 - t)^{5/2} |d_j^2 v(M_j)|^2 dt \cdot \int_0^1 (1 - t)^{-1/2} dt\right) r dr dz$$

$$= 6M_1^2 h^{-2} \sum_j \int_0^1 dt \int_c (1 - t)^{5/2} |[(r_j - r)\frac{\partial}{\partial r} + (z_j - z)\frac{\partial}{\partial z}]^2 v(M_j)|^2 r dr dz$$

$$\le 6M_1^2 h^{-2} \sum_j \int_0^1 dt \int_c (1 - t)^{5/2} h^4 \left(|\frac{\partial^2 v(M_j)}{\partial r \partial z}| + 2|\frac{\partial^2 v(M_j)}{\partial r \partial z}| + |\frac{\partial^2 v(M_j)}{\partial z^2}|\right)^2 r dr dz$$

where $M_2 = 72 M_1^2$. Make variable transformations in the integrals as follows:

$$\zeta = r_j t + r(1-t) \ , \quad \eta = z_j + z(1-t)$$

Then $M_j = (\zeta, \eta)$, and the triangle C reduces to a similar triangle $C_{j,t}$ with the similarity transformation center P_j. Hence the right side of (5.11) becomes:

$$M_2h^2 \sum_j \int_0^1 dt \int_{C_{j,t}} (1-t)^{-1/2} (\zeta - r_j t)\left(\left|\frac{\partial^2 v(\zeta,\eta)}{\partial \zeta^2}\right|^2 + \left|\frac{\partial^2 v(\zeta,\eta)}{\partial \zeta \partial \eta}\right|^2 + \left|\frac{\partial^2 v(\zeta,\eta)}{\partial \eta^2}\right|^2\right) d\zeta d\eta$$

$$\leq M_2h^2 \sum_j \int_0^1 dt \int_{C_{j,t}} (1-t)^{-1/2} \zeta(---) d\zeta d\eta \quad (\text{since } \zeta - r_j t \leq \zeta)$$

$$\leq M_2h^2 \sum_j \int_0^1 dt \int_C (1-t)^{-1/2} (---) \zeta d\zeta d\eta \quad (\text{since } C_{j,t} \subset C)$$

$$= 3M_2h^2 \int_C (---) \zeta d\zeta d\eta \cdot \int_0^1 (1-t)^{-1/2} dt \leq M_3 h^2 \| v \|^2_{2,c}$$

Hence we obtain by (5.11) that

$$\int_c \left(\frac{\partial v_I}{\partial r} - \frac{\partial v}{\partial r}\right)^2 r dr dz \leq M_3 h^2 \| v \|^2_{2,c} .$$

Similarly we obtain

$$\int_c \left(\frac{\partial v_I}{\partial z} - \frac{\partial v}{\partial z}\right)^2 r dr dz \leq M_4 h^2 \| v \|^2_{2,c} ,$$

$$\int_c (v_I - v)^2 r dr dz \leq M_5 h^2 \| v \|^2_{2,c} .$$

Therefore,

$$\| v - v_I \|^2_{1,c} \leq M h^2 \| v \|^2_{2,c} \quad \forall\, v \in C^\infty(c), \tag{5.12}$$

Finally (5.5) is deduced from (5.12) and lemma 3.3. Q.E.D.

Define "energy norm" $B_h(u,u)$ on Ω_h as follows:

$$B_h(u,u) = \int_{\Omega_h} \beta\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] r dr dz$$

Theorem 5.2. Assume that u_h is the solution of (5.1), u the solution of (4.1). Then

$$B_h(u-u_h, u-u_h) = O(h^2) \tag{5.13}$$

Proof: u_h minimizes the error $u - u_h$ in the "energy norm" on Ω — $B(v,v)$, i.e. (see [15, p. 39])

$$B(u - u_h, u - u_h) = \min_{v_h \subset V^h} B(u - v_h, u - v_h) .$$

Since $u_h = v_h = 0$ on $\Omega - \Omega_h$, we have

$$B_h(u - u_h, u - u_h) = \min_{v_h \in V^h} B_h(u - v_h, u - v_h) .$$

Define $u_I = 0$ on $\Omega - \Omega_h$. Then $u_I \in V^h$. So

$$B_h(u - u_h, u - u_h) \leq B_h(u - u_I, u - u_I) \leq \max_{\Omega} \beta \cdot \| u - u_I \|^2_{1,\Omega_h} \tag{5.14}$$

By virtue of lemma 5.1 we have

$$\| u - u_I \|^2_{1,\Omega_h} = \sum_{i=1}^{m} \| u - u_I \|^2_{1,c_i} \leq Mh^2 \sum_{i=1}^{m} \| u \|^2_{2,c_i} \leq Mh^2 \| u \|^2_{2,\Omega} \tag{5.15}$$

(5.14) and (5.15) prove that (5.13) is valid. Q.E.D.

If Ω is a polygon, then $\Omega_h = \Omega$, $B_h(v,v) = B(v,v)$. Since $B(u,v)$ is coercive on $V_0^1(\Omega)$, we have

Corollary 5.1. If Ω is a polygon, then

$$\| u - u_h \|_{1,\Omega} = O(h)$$

$$\| u - u_h \|_{0,\Omega} = O(h^2)$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The following model problem is studied:

$$\Omega \quad : -[\frac{1}{r}\frac{\partial}{\partial r}(r\beta\frac{\partial u}{\partial r}) + \frac{\partial}{\partial z}(\beta\frac{\partial u}{\partial z})] = f$$

$$\Gamma_1 \quad : u = 0$$

ABSTRACT (cont.)

where Ω is a bounded open domain with $r < 0$ in (r,z) plane, $\Gamma_1 = \partial\Omega \setminus \Gamma o$, $\Gamma o = \partial\Omega \cap \{(r,z) : r = 0\}$. We introduce weighted Sobolev spaces $V^k(k = 1,2)$, and prove:

(1) The problem has a unique solution u, and $u \in V_o^1(\Omega) \cap V^2(\Omega)$.

(2) The linear finite element solution u_h exists and is unique.

(3) The error $u-u_h$ in "energy norm" is of $0(h^2)$. Particularly, if Ω is a polygon, then

$$\|u - u_h\|_{1,\Omega} = 0(h)$$

$$\|u - u_h\|_{0,\Omega} = 0(h^2)$$

where $\|\cdot\|_{k,\Omega}(k = 1,2)$ are the V^k norms.


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